Pre Calc. BC   Abstract Algebra – REVIEW   Steve

1. Consider the relation $(\mathbb{Q}^*,*)$ where $a*b=\frac{a}{b}$ ($\mathbb{Q}^*$ is rational numbers without zero). Discuss the truth of the following statement:

   "Each element is its own inverse for this operation."

   If the identity were 1, this would be true. Sadly, there is no identity for this operation. It is true that $a*1=a$, but $1*a\neq a$. The identity must be commutative. ☹

2. Consider the operation $*$ such that $x*y=\frac{xy}{2}$.

   a) For which of the following sets is $*$ closed?

i) $\mathbb{Z}$	ii) $\mathbb{Q}^+$	iii) $\mathbb{R}$	iv) { {0, 1, 2}
no	yes	yes	no
ex $1*3\notin\mathbb{Z}$			$1*1\notin S$

   b) Is the operation associative? Demonstrate it.

   YES! $(a*b)*c \stackrel{?}{=} a*(b*c)$

   $\frac{ab}{2}*c = a*\frac{bc}{2}$

   $\frac{abc}{4}=\frac{abc}{4}$ ✓

   c) Is there an identity? Find it.

   YES! $a*e=a$

   $\frac{ae}{2}=a$

   $e=2$

   note: if $a=0$ 2 still works! $0*2=0$

   and it commutes! $a*2=2*a$

   d) What is the inverse of 5?

   $5*5^{-1}=2$

   $\frac{5\cdot5^{-1}}{2}=2$ → $5\cdot5^{-1}=4$

   $5^{-1}=4/5$

   e) Is ({2, 4, 6, 8, ...}, $*$) an abelian group? Explain.

   NO

   assoc ✓

   comm ✓

   id = 2, in group ✓

   closed ✓ $\left(\frac{(2n)(2m)}{2}=2mn\in\text{evens}\right)$

   BUT there is a problem with inverses ☹

   ex. $6^{-1}=\frac{2}{3}\notin S$

3. Determine which (if any) relations below form abelian groups. Demonstrate **all** the properties. Be clear, concise, and complete.

a) $(\mathbb{Z}^*_{39}, \otimes)$

✓ id = 1

✓ assoc ex $(2 \otimes 3) \otimes 4 = 24 = 2 \otimes (3 \otimes 4)$

not closed! ex $3 \otimes 13 = 0 \notin \mathbb{Z}^*_{39}$

and not all elements have inverses ex. 3^{-1} DNE

NO!

b) $(\mathbb{Z}, *)$ *where* $x * y = |x - y|$

✓ closed ex $3 * 5 = 2$

✓ comm ex $3 * 5 = 5 * 3$

X id = 0? $a * 0 = a$ only if $a \geq 0$

X assoc? NO. $(3 * 5) * 11 \overset{?}{=} 3 * (5 * 11)$
$= 2 * 11 \quad 3 * 6$
$= 9 \quad = 3$

inverses? $a^{-1} = a$ but no identity so never mind.

c)

°	!	#	$	%
!	!	%	!	#
#	%	#	#	$
$	!	#	$	%
%	#	$	%	!

✓ id = $

✓ com. (reflects over diagonal)

X inverses there is no inverse for ! ☹

X and not assoc ex. $! \circ (\# \circ \#) = \%$ but $(! \circ \#) \circ \# = \$$

d) $(\sqrt[4]{-1}, \times)$

$\left\{ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right\}$

✓ com

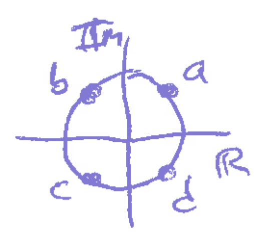
✓ assoc

✓ inv

but no identity!

and not closed!

ex $a \times b = -1 \notin$ Set

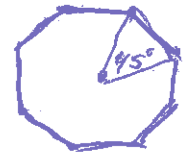


4. Let H be the set of symmetry *rotations* for a regular octagon.

a) Make a list of the elements of H. Indicate the order of each element.

$\{R_0, R_{45}, R_{90}, R_{135}, R_{180}, R_{225}, R_{270}, R_{315}\}$

1 8 4 8 2 8 4 8



b) Is (H, ∗) a group?

Yes.

closed ✓ assoc ✓ comm ✓ id = R_0 ✓ inverses exist ✓

c) List all the subgroups of H.

$\{R_0, R_{90}, R_{180}, R_{270}\}$

$\{R_0, R_{180}\}$

d) Determine which groups below are isomorphic to H. Clearly write out elements and element orders for each group.

sym(□)

$\{R_0, R_{90}, R_{180}, R_{270}, r_l, r_m, r_n, r_p\}$

1 4 2 4 2 2 2 2

$\left(\left\{R_{0°}, R_{90°}, R_{180°}, R_{270°}, r_x, r_y, r_{y=x}, r_{y=-x}\right\}, \circ\right)$

1 4 2 4 2 2 2 2

$(\mathbb{Z}_8, \oplus)$

$\{0, 1, 2, 3, 4, 5, 6, 7\}$

1 8 4 8 2 8 4 8

$\left(\left\{\sqrt[8]{1}\right\}, \times\right)$

↓

$\{1, 1\text{cis}45, 1\text{cis}90, 1\text{cis}135, 1\text{cis}180, 1\text{cis}225, 1\text{cis}270, 1\text{cis}315\}$

1 8 4 8 2 8 4 8

$\left(\left\{1, 3, 5, 7, 9, 11, 13, 15\right\}, \otimes_{16}\right)$

1 4 4 2 2 4 4 2

$$\left(\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\right\}, \times\right)$$

1 2 2 2 2 2 2 2

5. Five groups are defined below. Determine which are isomorphic to which.

a) $(\mathbb{Z}_4, \oplus)$

{0, 1, 2, 3}
1 4 2 4

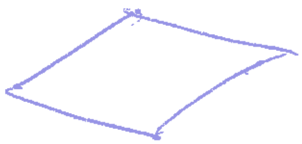
b) $(\mathbb{Z}_5^*, \otimes)$

1, 2, 3, 4
1 4 4 2

c)

	2	1	2	2
#	**a**	**b**	**c**	**d**
a	b	a	d	c
b	a	b	c	d
c	d	c	b	a
d	c	d	a	b

d) The symmetry group for a rhombus.



$\{R_0, R_{180}, r_h, r_v\}$
1 2 2 2

e) The *rotational* symmetry group of a square.

$\{R_0, R_{90}, R_{180}, R_{270}\}$
1 4 2 4

f) $(\{1, i, -1, -i\}, \times)$

1 4 2 4

g) $\left(\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right\}, \times\right)$

1 2 2 2

a, b, e, f

c, d, g

6. List all possible subgroups for $(\mathbb{Z}_{18}, \oplus)$

only possible orders are 2, 3, 6, 9

{0, 9}

{0, 2, 4, 6, 8, 10, 12, 14, 16}

{0, 6, 12}

{0, 3, 6, 9, 12, 15}

7. Consider the statements below and provide a reason, example or counter example for each.

a) *A non-cyclic group can have a cyclic sub-group.*

True. Ex. sym(□) is non-cyclic, but rotation subgroup is cyclic.

b) *A cyclic group can have a non-cyclic subgroup*

False, Suppose a, b are 2 elements in H w/ no common generator → they have no common generator in G either

c) *A non-abelian group can have an abelian sub-group.*

True, same example as (a)

d) *An abelian group can have a non-abelian sub-group.*

False. if $a*b \neq b*a$ in H, then they won't be equal in G either

8. Consider the group $(\mathbb{Z}_7^*, \otimes)$

a) List all the elements with their orders.

1	2	3	4	5	6
(1)	(3)	(6)	(3)	(6)	(2)

b) What is the inverse of 4?

$4^{-1} = 3$

c) List a generator of $(\mathbb{Z}_7^*, \otimes)$.

$3 \to 2 \to 6 \to 4 \to 5 \to 1$

d) Find a subgroup of order 3.

$\{1, 2, 4\}$

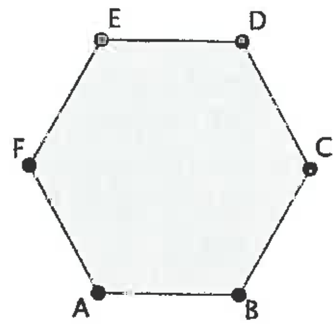
e) Explain why $(\mathbb{Z}_7, \oplus)$ has no subgroups.

$o(\mathbb{Z}_7) = 7$, which has no factors, thus all elements (aside from 0) must be order 7, ie. generators. A subgroup can never contain generators!

9. Consider the symmetry group, H, for the regular hexagon shown at below.

a) Draw or describe one transformation that is an element of H. (*NOT the identity).*

$R_{60°}$



b) What is the order of H? (*hint: the order of the symmetry group for a regular n-gon is always 2n).*

$o(H) = 12$

identity, 5 rotations, 6 reflections

c) Compare H to $(\mathbb{Z}_{12}, \oplus)$. Explain why these two groups are not isomorphic.

$\mathbb{Z}_{12}$ is cyclic, the elements 1, 5, 7, 11 are all generators. H is not cyclic.

d) Find two isomorphic subgroups from each of these groups.

$\{R_0, R_{60°}, R_{120°}, R_{180}, R_{240°}, R_{300}\}$

isomorphic to $\{0, 2, 4, 6, 8, 10\}$

10. Given the matrix M = $\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}$.

a) Find the multiplicative group generated by M.

$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ i & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) Is this group abelian?

Yes

c) Find a subgroup of this group.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$

d) Find the inverse of M.

$M^{-1} = \begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}$

e) Explain why no subgroup could contain M.

It is a generator

Steve

Pre Calc BC   Matrices and Groups Review

All except M have orders 1, 2, 4, 4

1. Which of the five groups below are isomorphic?

G=$\left(z_5^*, \otimes\right)$; H=$\left(z_4, \oplus\right)$; J=square rotation group

K=$\left(\{1, -1, i, -i\}, \times\right)$ M=rectangle symmetry group

a) G~H only   b) H~J only   c) K~M only
d) H~J~K only   e) G~H~J~K

If $a=a^{-1}$ then $o(a)=2$, but 2 is not a factor of 15.

2. Let a be an element of group G, a group of order 15. Which of the statements below must not be true?

a) a is the identity   b) a is its own inverse (but not the identity)

c) $a^4 = a$ d) a is in a subgroup of G
e) a is not in a subgroup of G

The inverses are not all in set. Also identity not in set.

3. Consider the operaton $*$ such that $a * b = 2ab$. Which of these is false?

a) there is an identity for $*$ b) $*$ is associative
c) $*$ is commutative
d) $*$ is closed for even numbers   e) $*$ forms a group with the even numbers

4. Which of these is a cyclic group?

a) $\left(z_7^*, \otimes\right)$
b) the symmetry group for a square
c) the symmetry group for an equilateral Δ
d) the permutation group of four elements
e) the permutation group of 5 elements

5. Given $\begin{bmatrix} 2 & 3 & a \\ b & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$

the sum of a and b is

a) 1 b) -1 c) 4 d) -3 e) -4

$2+15+2a=15 \to a=-1$
$b+5+8=10 \to b=-3$
$a+b=-4$

6. The transformation matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ represents

a) $r_{y=x}$ b) $r_{y=-x}$ c) $R_{180°}$
d) $R_{90°}$ e) $R_{-90°}$

$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

7. Given the system of equations:

$$\begin{cases} x - 3y + z = -2 \\ 2x + 3y - 4z = -4 \\ x + y - z = 0 \end{cases}$$

use $\begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} \to x=1, y=2, z=3$

the product xyz equals

a) 0   b) 6   c) -3   d) 10   e) -12

8. If $k = \dfrac{453!}{450!3!}$, then   *$k = \dfrac{453 \cdot 452 \cdot 451}{6}$*

a) $k > 10^{100}$ b) $10^{10} \le k < 10^{100}$
c) $10^5 \le k < 10^{10}$ d) $10 \le k < 10^5$
e) $k < 10$

9. (p,q) is called a *lattice point* if p and q are both integers. How many lattice points lie in the area between the two curves $x^2 + y^2 = 9$ and $x^2 + y^2 - 6x + 5 = 0$?

a) 0   b) 1   c) 2   d) 3   e) 4

10. If the determinant of the matrix $\begin{bmatrix} 7 & a \\ 4 & 3 \end{bmatrix} = 1$

then a must equal

a) -1   b) 0   c) 1   d) 2   e) 5

$21-4a=1$, $a=5$

11. If the matrix $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ is in a multiplicative group, which of these must also be in the group?

I. $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ **II.** $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ **III.** $\begin{bmatrix} 0 & 0.5 \\ -1 & 0 \end{bmatrix}$

a) I only   b) II only   c) III only
d) I and II   e) II and III

$A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$
$A^{-1} = \begin{bmatrix} 0 & 0.5 \\ -1 & 0 \end{bmatrix}$
$A^2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$