Every limited straight line can be extended indefinitely to a (unique) straight line. [Appendix A]

In addition, Euclid defines a right angle as follows:

When a straight line intersects another straight line such that the adjacent angles are equal to one another, then the equal angles are called ***right angles***. [Appendix A]

Note that if you use this definition, then right angles at a cone point are not equal to right angles at points where the cone is locally isometric to the plane. And Euclid goes on to state as his Fourth Postulate:

All right angles are equal.

Thus, Euclid's Fourth Postulate rules out cones and any surface with isolated cone points. What is further ruled out by Euclid's Fourth Postulate would depend on formulating more precisely just what it says. It is not clear (at least to the author!) whether there is something we would want to call a surface that could be said to satisfy Euclid's Fourth Postulate and not be a smooth surface. However, it is clear that Euclid's postulate at least gives part of the meaning of "smooth surface," because it rules out isolated cone points.

Chapter 5

Straightness on Hyperbolic Planes

[To son János:] For God's sake, please give it [work on hyperbolic geometry] up. Fear it no less than the sensual passion, because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life.
— Wolfgang Bolyai (1775–1856)
[**SE**: Davis and Hersch, page 220]

We now study hyperbolic geometry. This chapter may be skipped if the reader will not be covering geometric manifolds and the shape of space in Chapters 17 and 22 and if in the remainder of this book the reader leaves out all mentions of hyperbolic planes. However, to skip studying hyperbolic planes would be to skip an important notion in the history of geometry, and to skip the geometry which may be the basis of our physical universe.

As with the cone and cylinder, we must use an intrinsic point of view on hyperbolic planes. This is especially true because there is no standard extrinsic embedding of a hyperbolic plane into 3-space.

A Short History of Hyperbolic Geometry

Hyperbolic geometry, discovered more than 170 years ago by C.F. Gauss (1777–1855, German), János Bolyai (1802–1860, Hungarian), and N.I. Lobatchevsky (1792–1856, Russian), is special from a formal axiomatic point of view because it satisfies all the postulates (axioms) of Euclidean geometry except for the parallel postulate. In hyperbolic geometry straight lines can converge toward each other without intersecting (violating Euclid's Fifth Postulate), and there is more than one straight line through a given point that does not intersect (is parallel to) a given line (violating Playfair's Parallel Postulate). (See Figure 5.1.)

The reader can explore more details of the axiomatic nature of hyperbolic geometry in Chapter 10. Note that the 450° cone also violates the two parallel postulates mentioned above. Thus the 450° cone has many of the properties of the hyperbolic plane.

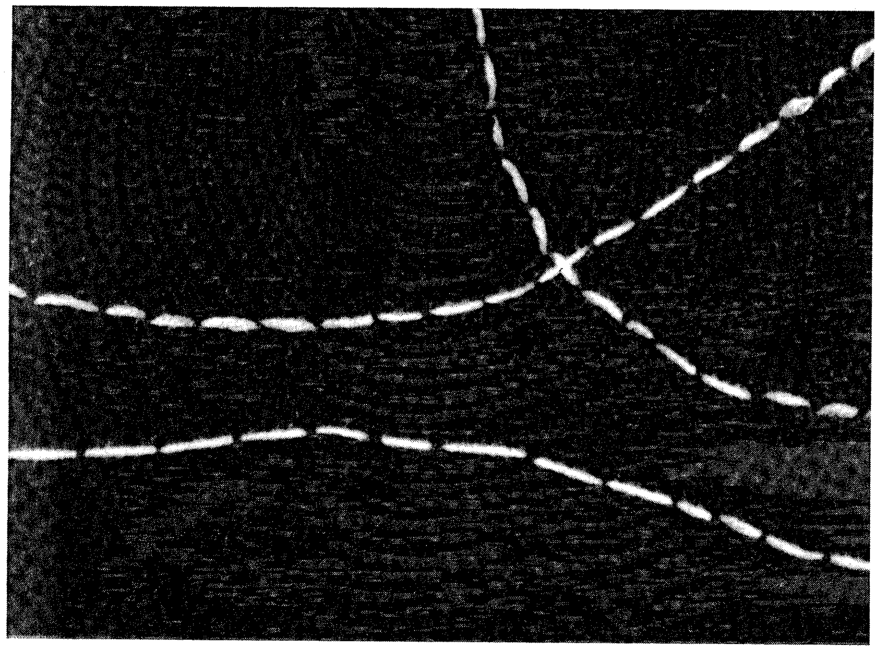


Figure 5.1 Two geodesics through a point not intersecting a given line

Hyperbolic geometry has turned out to be useful in various branches of higher mathematics. Also, the geometry of binocular visual space appears experimentally to be best represented by hyperbolic geometry (see [**NE**: Zage]). In addition, hyperbolic geometry is one of the possible geometries for our three-dimensional physical universe — we will explore this connection more in Chapters 17 and 22.

Hyperbolic geometry and non-Euclidean geometry are considered in many books as being synonymous, but as we have seen there are other non-Euclidean geometries, particularly spherical geometry. It is also not accurate to say (as many books do) that non-Euclidean geometry was discovered about 170 years ago. Spherical geometry (which is clearly

not Euclidean) was in existence and studied by at least the ancient Babylonians, Indians, and Greeks more than 2,000 years ago. Spherical geometry was of importance for astronomical observations and astrological calculations. In Aristotle we can find evidence that non-Euclidean geometry was studied even before Euclid. (See [**Hi**: Heath, page 57] and [**Hi:** Toth].) Even Euclid in his *Phaenomena* [**AT**: Euclid] (a work on astronomy) discusses propositions of spherical geometry. Menelaus, a Greek of the first century, published a book *Sphaerica*, which contains many theorems about spherical triangles and compares them to triangles on the Euclidean plane. (*Sphaerica* survives only in an Arabic version. For a discussion see [**Hi**: Kline, page 119–120].)

Most texts and popular books introduce hyperbolic geometry either axiomatically or via "models" of the hyperbolic geometry in the Euclidean plane. These models are like our familiar map projections of the earth and (like these maps of the earth) intrinsic straight lines on the hyperbolic plane (surface of the earth) are not, in general, straight in the model (map) and the model, in general, distorts distances and angles. We will return to the subject of projection and models in Chapter 16.

In this chapter we will introduce the geometry of the hyperbolic plane as the intrinsic geometry of a particular surface in 3-space, in much the same way that we introduced spherical geometry by looking at the intrinsic geometry of the sphere in 3-space. Such a surface is called an ***isometric embedding*** of the hyperbolic plane into 3-space. We will construct such a surface in the next section. Nevertheless, many texts and popular books say that David Hilbert (1862–1943, German) proved in 1901 that it is not possible to have an isometric embedding of the hyperbolic plane onto a closed subset of Euclidean 3-space. These authors miss what Hilbert actually proved. In fact, Hilbert [**NE**: Hilbert] proved that there is no ***real analytic*** isometry (that is, no isometry defined by real-valued functions which have convergent power series). In 1972, Tilla Milnor [**NE**: Milnor] extended Hilbert's result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. Without giving an explicit construction, N. Kuiper [**NE**: Kuiper] showed in 1955 that there is a differentiable isometric embedding onto a closed subset of 3-space.

The construction used here was shown to the author by William Thurston (b.1946, American) in 1978†; and it is not defined by equations

†The idea for this construction is also included in Thurston's recent book [**DG:** Thurston, pages 49 and 50] and is discussed in the author's recent book [**DG:** Henderson, page 31].

at all, because it has no definite embedding in Euclidean space. In Problem **5.2** we will show that our isometric model is locally isometric to a certain smooth surface of revolution called the ***pseudosphere***, which is well known to locally have hyperbolic geometry. Later, in Chapter 16, we will explore the various (non-isometric) models of the hyperbolic plane (these models are the way that hyperbolic geometry is presented in most texts) and prove that these models and the isometric constructions here produce the same geometry.

Constructions of Hyperbolic Planes

We will describe four different isometric constructions of hyperbolic planes (or approximations to hyperbolic planes) as surfaces in 3-space. It is very important that you actually perform at least one of these constructions. The act of constructing the surface will give you a feel for hyperbolic planes that is difficult to get any other way. Templates for all the paper constructions (and information about possible availability of crocheted hyperbolic planes) can be found at the supplements site

www.math.cornell.edu/~dwh/books/eg00/supplements.html

1. The Hyperbolic Plane from Paper Annuli

A paper model of the hyperbolic plane may be constructed as follows: Cut out many identical annular ("annulus" is the region between two concentric circles) strips as in Figure 5.2. Attach the strips together by taping the inner circle of one to the outer circle of the other. It is crucial that all the annular strips have the same inner radius and the same outer radius, but the lengths of the annular strips do not matter. You can also cut an annular strip shorter or extend an annular strip by taping two strips together along their straight ends. The resulting surface is of course only an approximation of the desired surface. The actual hyperbolic plane is obtained by letting $\delta \to 0$ while holding the radius ρ fixed. Note that since the surface is constructed (as $\delta \to 0$) the same everywhere it is ***homogeneous*** (that is, intrinsically and geometrically, every point has a neighborhood that is isometric to a neighborhood of any other point). We will call the results of this construction the ***annular hyperbolic plane.*** I strongly suggest that the reader take the time to **cut out carefully several such annuli and tape them together as indicated.**

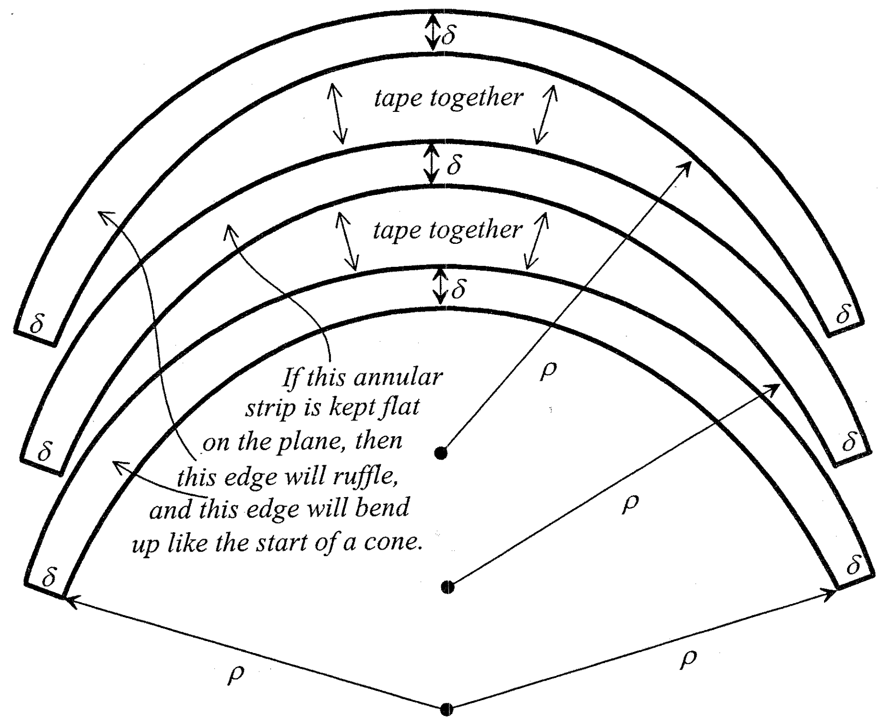


Figure 5.2 Annular strips for making an annular hyperbolic plane

2. How to Crochet the Hyperbolic Plane

Once you have tried to make your annular hyperbolic plane from paper annuli you will certainly realize that it will take a lot of time. Also, later you will have to play with it carefully because it is fragile and tears and creases easily — you may want just to have it sitting on your desk. But there is another way to get a sturdier model of the hyperbolic plane, which you can work and play with as much as you wish. This is the crocheted hyperbolic plane.

In order to make the crocheted hyperbolic plane you need just very basic crocheting skills. All you need to know is how to make a chain (to start) and how to single crochet. That's it! Now you can start. See Figure 5.3 for a picture of these stitches, and see their description in the list below.

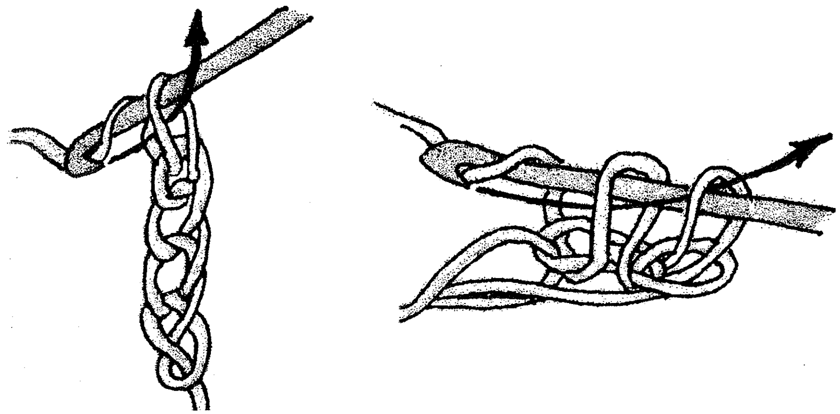


Figure 5.3 Crochet stitches for the hyperbolic plane

First you should choose a yarn that will not stretch a lot. Every yarn will stretch a little but you need one that will keep its shape. Now you are ready to start the stitches:

1. Make your **beginning chain stitches** (Figure 5.3a). About 20 chain stitches for the beginning will be enough.
2. **For the first stitch in each row** insert the hook into the 2nd chain from the hook. Take yarn over and pull through chain, leaving 2 loops on hook. Take yarn over and pull through both loops. One single crochet stitch has been completed. (Figure 5.3b.)
3. **For the next N stitches** proceed exactly like the first stitch except insert the hook into the next chain (instead of the 2nd).
4. **For the $(N + 1)$st stitch** proceed as before except insert the hook into the same loop as the N-th stitch.
5. **Repeat Steps 3 and 4** until you reach the end of the row.
6. **At the end of the row** before going to the next row do one extra chain stitch.
7. **When you have the model as big as you want**, you can stop by just pulling the yarn through the last loop.

Be sure to crochet fairly tightly and evenly. That's all you need from crochet basics. Now you can go ahead and make your own

hyperbolic plane. You have to increase (by the above procedure) the number of stitches from one row to the next in a constant ratio, N to N + 1 — the ratio and size of the yarn determine the radius (the ρ in the annular hyperbolic plane) of the hyperbolic plane. You can experiment with different ratios BUT not in the same model. We suggest that you start with a ratio of 5 to 6. You will get a hyperbolic plane ONLY if you will be increasing the number of stitches in the same ratio all the time.

Crocheting will take some time but later you can work with this model without worrying about destroying it. The completed product is pictured in Figure 5.4.

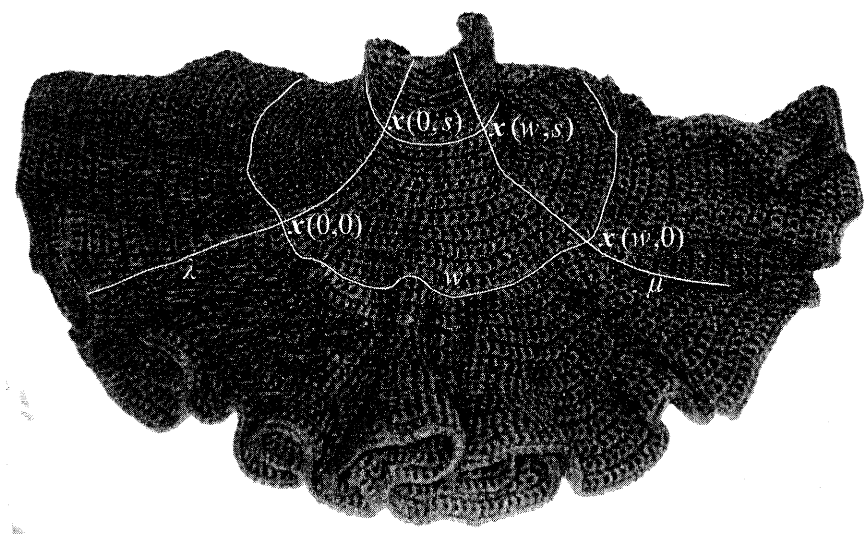


Figure 5.4 A crocheted annular hyperbolic plane

3. {3,7} and {7,3} Polyhedral Constructions

A polyhedral model can be constructed from equilateral triangles by putting 7 triangles together at every vertex, or by putting 3 regular heptagons (7-gons) together at every vertex. These are called the **{3,7} polyhedral model** and the **{7,3} polyhedral model** because triangles (3-gons) are put together **7** at a vertex, or heptagons (7-gons) are put together 3 at a vertex. These models have the advantage of being constructed more easily than the annular or crocheted models; however, one cannot make better and better approximations by decreasing the size of the triangles. This is true because at each vertex the cone angle is $(7 \times \pi/3) = 420°$ or $(3 \times 5\pi/7) = 385.71...°$), no matter what the size of the triangles and heptagons are; whereas the hyperbolic plane in the small looks like the Euclidean plane with 360° cone angles. Another disadvantage of the polyhedral model is that it is not easy to describe the annuli and related coordinates.

You can make these models less "pointy" by replacing the sides of the triangles with arcs of circles in such a way that the new vertex angles are $2\pi/7$, or by replacing the sides of the heptagons with arcs of circles in such a way that the new vertex angles are $2\pi/3$. But then the model is less easy to construct because you are cutting and taping along curved edges.

See Problems **10.6** and **21.5** for more discussions of regular polyhedral tilings of plane, spheres, and hyperbolic planes.

4. Hyperbolic Soccer Ball Construction

We now explore a polyhedral construction that involves two different regular polygons instead of the single polygon used in the {3,7} and {7,3} polyhedral constructions. A spherical soccer ball (outside the USA, called a football) is constructed by using pentagons surrounded by five hexagons or two hexagons and one pentagon together around each vertex. The plane can be tiled by hexagons, each surrounded by six other hexagons. The hyperbolic plane can be approximately constructed by using heptagons (7-sided) surrounded by seven hexagons and two hexagons and one heptagon together around each vertex. See Figure 5.5. Because a heptagon has interior angles with $5\pi/7$ radians (= 128.57...°), the vertices of this construction have cone angles of 368.57...° and thus are much smoother than the {3,7} and {7,3} polyhedral constructions. It also has a nice appearance if you make the heptagons a different color from the hexagons. It is also easy to construct (as long as you have a template — you can find a variety on the supplements website). As with any polyhedral construction one cannot get closer and closer approximations to the hyperbolic plane. There is also no apparent way to see the annuli.

The hyperbolic soccer ball construction is related to the {3,7} construction in the sense that if a neighborhood of each vertex in the {3,7} construction is replaced by a heptagon then the remaining portion of each triangle is a hexagon.

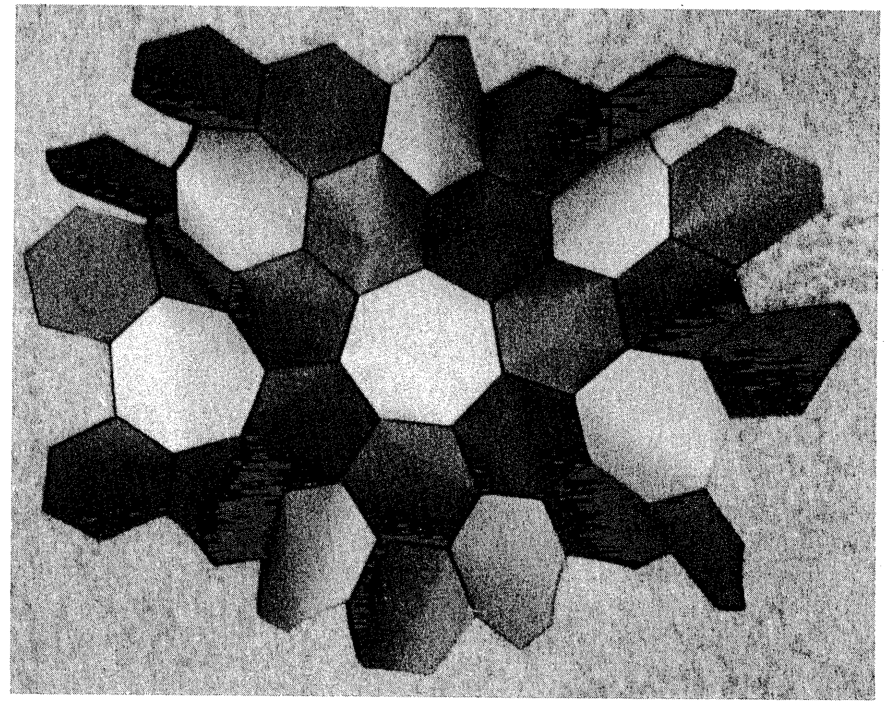


Figure 5.5 The hyperbolic soccer ball

5. "{3,6½}" Polyhedral Construction

We can avoid some of the disadvantages of the {3,7} and soccer ball constructions by constructing a polyhedral annulus. In this construction we have seven triangles together only at every other vertex and six triangles together at the others. This construction still has the disadvantage of not being able to produce closer and closer approximations and it also is more "pointy" (larger cone angles) than the hyperbolic soccer ball.

The precise construction can be described in two different (but, in the end, equivalent) ways:

1. Construct polyhedral annuli as in Figure 5.6 and then tape them together as with the annular hyperbolic plane.

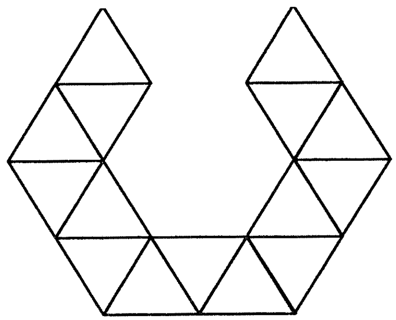


Figure 5.6 Polyhedral annulus

2. The quickest way is to start with many strips as pictured in Figure 5.7a — these strips can be as long as you wish. Then add four of the strips together as in Figure 5.7b, using five additional triangles. Next, add another strip every place there is a vertex with five triangles and a gap (as at the marked vertices in Figure 5.7b). Every time a strip is added an additional vertex with seven triangles is formed.

The center of each strip runs perpendicular to each annulus, and you can show that these curves (the center lines of the strip) are each geodesics because they have local reflection symmetry.

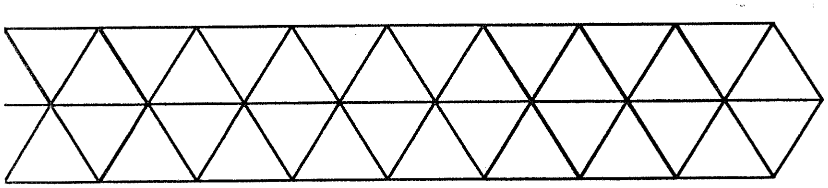


Figure 5.7a Strips

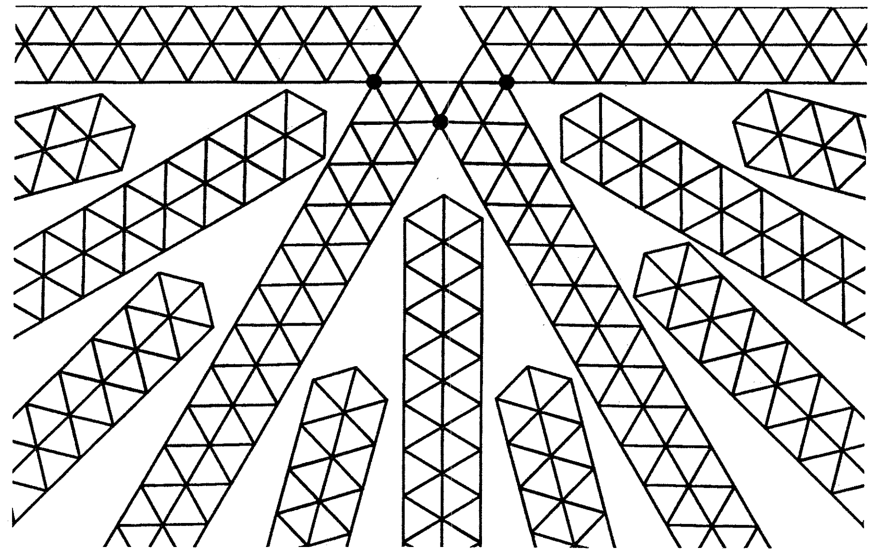


Figure 5.7b Forming the polyhedral annular hyperbolic plane

Hyperbolic Planes of Different Radii (Curvature)

Note that the construction of a hyperbolic plane is dependent on ρ (the radius of the annuli), which is often called the ***radius of the hyperbolic plane***. As in the case of spheres, we get different hyperbolic planes depending on the value of ρ. In Figures 5.8–5.10 (see next page) there are crocheted hyperbolic planes with radii approximately 4 cm, 8 cm, and 16 cm. The pictures were all taken from approximately the same perspective and in each picture there is a centimeter rule to indicate the scale.

Note that as ρ increases the hyperbolic plane becomes flatter and flatter (has less and less curvature). For both the sphere and the hyperbolic plane as ρ goes to infinity they both become indistinguishable from the ordinary flat (Euclidean) plane. Thus, the plane can be called a sphere (or hyperbolic plane) with infinite radius. In Chapter 7, we will define the "Gaussian Curvature" and show that it is equal to $1/\rho^2$ for a sphere and $-1/\rho^2$ for a hyperbolic plane.

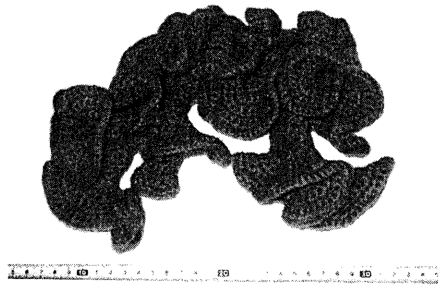


Figure 5.8 Hyperbolic plane with $\rho \approx 4$ cm

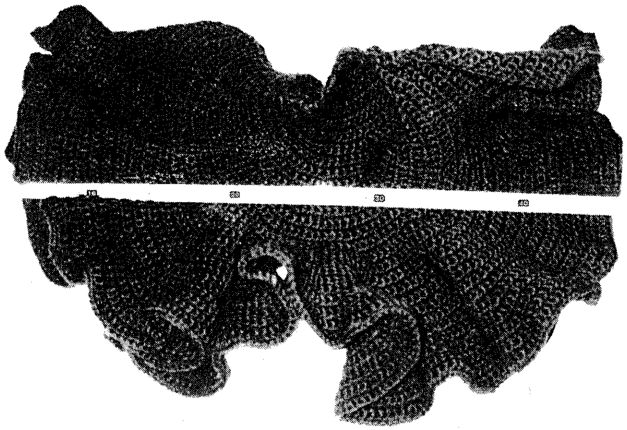


Figure 5.9 Hyperbolic plane with $\rho \approx 8$ cm

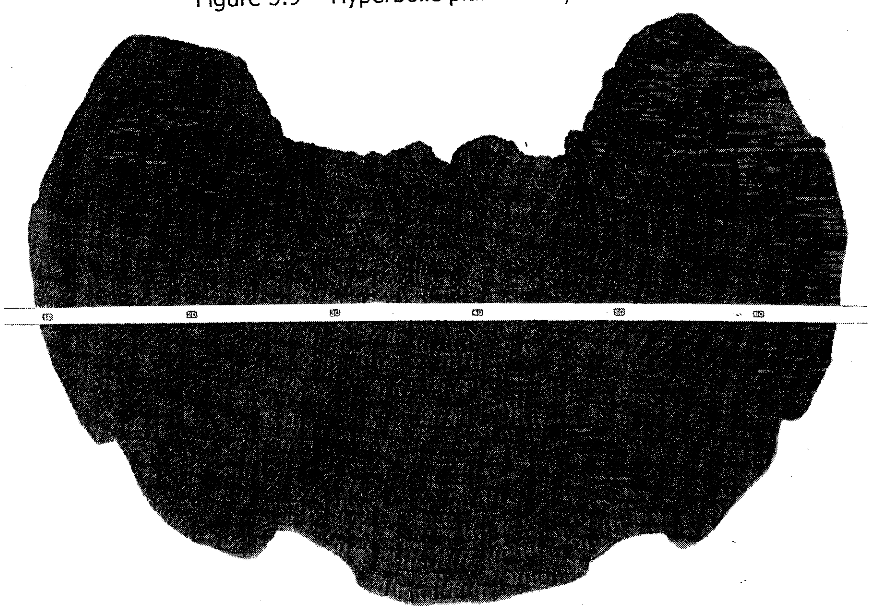


Figure 5.10 Hyperbolic plane with $\rho \approx 16$ cm

PROBLEM 5.1 WHAT IS STRAIGHT IN A HYPERBOLIC PLANE?

a. *On a hyperbolic plane, consider the curves that run radially across each annular strip. Argue that these curves are intrinsically straight. Also, show that any two of them are asymptotic, in the sense that they converge toward each other but do not intersect.*

Look for the local intrinsic symmetries of each annular strip and then global symmetries in the whole hyperbolic plane. Make sure you give a convincing argument why the symmetry holds in the limit as $\delta \to 0$.

We shall say that two geodesics that converge in this way are ***asymptotic geodesics***. Note that there are no geodesics (straight lines) on the plane that are asymptotic.

b. *Find other geodesics on your physical hyperbolic surface. Use the properties of straightness (such as symmetries) you talked about in Problems* **1.1**, **2.1**, *and* **4.1**.

Try holding two points between the index fingers and thumbs on your two hands. Now pull gently — a geodesic segment with its reflection symmetry should appear between the two points. If your surface is durable enough, try folding the surface along a geodesic. Also, you may use a ribbon to test for geodesics.

c. *What properties do you notice for geodesics on a hyperbolic plane? How are they the same as geodesics on the plane or spheres, and how are they different from geodesics on the plane and spheres?*

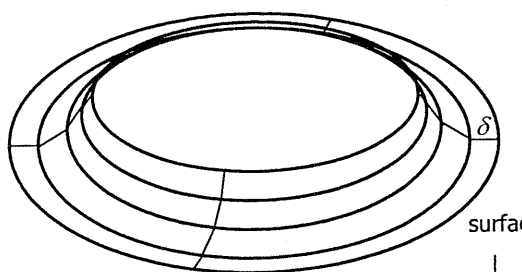
Explore properties of geodesics involving intersecting, uniqueness, and symmetries. Convince yourself as much as possible using your model — full proofs for some of the properties will have to wait until Chapter 16.

*PROBLEM 5.2 THE PSEUDOSPHERE IS HYPERBOLIC

Show that locally the annular hyperbolic plane is isometric to portions of a (smooth) surface defined by revolving the graph of *a continuously differentiable function of z about the z-axis. This is the surface usually called the* ***pseudosphere***.

Outline of Proof

1. Argue that each point on the annular hyperbolic plane is like any other point. (Think of the annular construction.)
2. Start with one of the annular strips and complete it to a full annulus in a plane. Then, construct a surface of revolution by attaching to the inside edge of this annulus other annular strips as described in the construction of the annular hyperbolic plane. (See Figure 5.11.) Note that the second and subsequent annuli form truncated cones. Finally, imagine the width of the annular strips, δ, shrinking to zero.



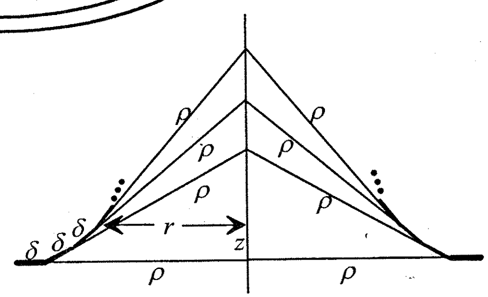


Figure 5.11 Hyperbolic surface of revolution — pseudosphere

3. Derive a differential equation representing the coordinates of a point on the surface using the geometry inherent in Figure 5.11. If $f(r)$ is the height (z-coordinate) of the surface at a distance of r from the z-axis, then the differential equation should be (remember that ρ is a constant)

$$\frac{dz}{dr} = \frac{-r}{\sqrt{\rho^2 - r^2}}$$

4. Solve (using tables or computer algebra systems) the differential equation for z as a function of r. Note that you are not getting r as a function of z.
5. Then argue (using a theorem from first-semester calculus) that r is a continuously differentiable function of z.

We can also crochet a pseudosphere by starting with 5 or 6 chain stitches and continuing in a spiral fashion, increasing as when crocheting the hyperbolic plane. See Figure 5.12. Note that, when you crochet beyond the annular strip that lays flat and forms a complete annulus, the surface forms ruffles and is no longer a surface of revolution (nor a smooth surface).

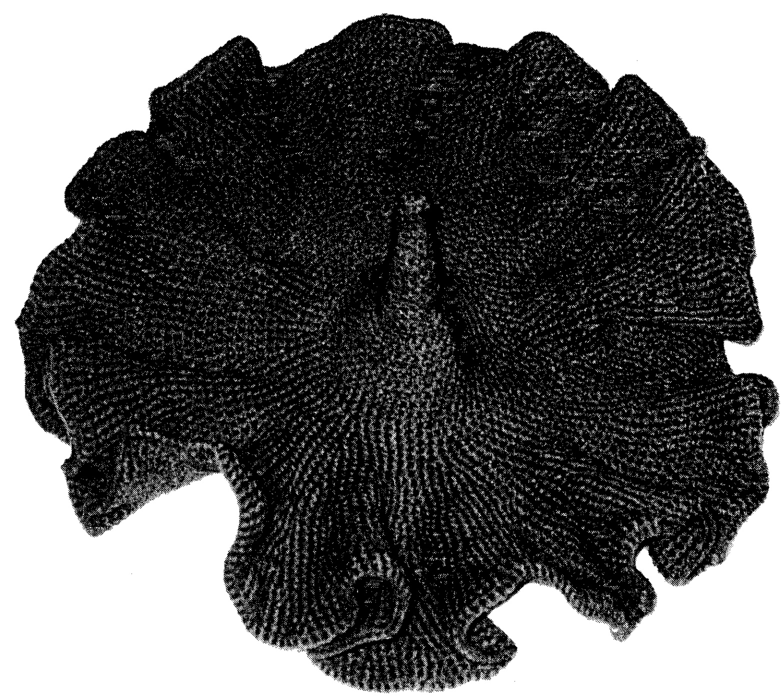


Figure 5.12 Crocheted pseudosphere