finished (planed) boards, but when we are close enough to touch it, we see that its surface is made up of smooth waves or ripples, and under a microscope we see the non-smoothness of numerous twisting fibers. See Figure 1.16.

Chapter 2

STRAIGHTNESS ON SPHERES

> ... it will readily be seen how much space lies between the two places themselves on the circumference of the large circle which is drawn through them around the earth. ... [W]e grant that it has been demonstrated by mathematics that the surface of the land and water is in its entirety a sphere, ... and that any plane which passes through the center makes at its surface, that is, at the surface of the earth and of the sky, great circles, and that the angles of the planes, which angles are at the center, cut the circumferences of the circles which they intercept proportionately, ...
>
> — Ptolemy, *Geographia* (ca. 150 AD) Book One, Chapter II

Drawing on the understandings about straightness you developed in Problem **1.1**, the second problem asks you to investigate the notion of straightness on a sphere. It is important for you to realize that, if you are not building a notion of straightness for yourself (for example, if you are taking ideas from books without thinking deeply about them), then you will have difficulty building a concept of straightness on surfaces other than a plane. Only by developing a personal meaning of straightness for yourself does it become part of your active intuition. We say *active* intuition to emphasize that intuition is in a process of constant change and enrichment, that it is not static.

PROBLEM 2.1 WHAT IS STRAIGHT ON A SPHERE?

a. *Imagine yourself to be a bug crawling around on a sphere. (This bug can neither fly nor burrow into the sphere.) The bug's universe is just the surface; it never leaves it. What is "straight" for this bug? What will the bug see or experience as straight? How can you convince yourself of this? Use the properties of* *straightness (such as symmetries) that you talked about in Problem* **1.1**.

b. *Show (that is, convince yourself, and give an argument to convince others) that the great circles on a sphere are straight with respect to the sphere, and that no other circles on the sphere are straight with respect to the sphere.*

SUGGESTIONS

Great circles are those circles which are the intersection of the sphere with a plane through the center of the sphere. Examples include any longitude line and the equator on the earth. Note that any pair of opposite points can be considered as the poles, and thus the equator and longitudes with respect to any pair of opposite points will be great circles. See Figure 2.1.

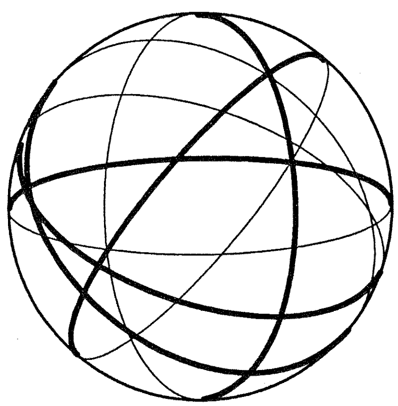

Figure 2.1 Great circles

The first step to understanding this problem is to convince yourself that great circles are straight lines on a sphere. Think what it is about the great circles that would make the bug experience them as straight. To better visualize what is happening on a sphere (or any other surface, for that matter), **you must use models**. This is a point we cannot stress enough. The use of models will become increasingly important in later problems, especially those involving more than one line. You must make lines on a sphere to fully understand what is straight and why. An orange or an old, worn tennis ball work well as spheres, and rubber bands make

good lines. Also, you can use ribbon or strips of paper. Try placing these items on the sphere along different curves to see what happens.

Also look at the symmetries from Problem **1.1** to see if they hold for straight lines on the sphere. The important thing here is to **think in terms of the surface of the sphere, not the solid 3-dimensional ball**. Always try to imagine how things would look from the bug's point of view. A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very 2-dimensional perception of the world around it — to the water strider, there is no up or down; its whole world consists of the 2-dimensional plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. Hungry birds and fish take advantage of this fact. This is the type of thinking needed to adequately visualize properties of straight lines on the sphere. For more discussion of water striders and other animals with their own varieties of intrinsic observations, see the delightful book, *The View from the Oak*, by Judith and Herbert Kohl [**Na**: Kohl and Kohl].

Lines that are straight on a sphere (or other surfaces) are often called ***geodesics***. This leads us to consider the concept of intrinsic or geodesic curvature versus extrinsic curvature. As an outside observer looking at the sphere in 3-space, all paths on the sphere, even the great circles, are curved — that is, they exhibit *extrinsic* curvature. But relative to the surface of the sphere (*intrinsically*), the lines may be straight. Be sure to understand this difference and to see why all symmetries (such as reflections) must be carried out intrinsically, or from the bug's point of view.

It is natural for you to have some difficulty experiencing straightness on surfaces other than the 2-dimensional plane; it is likely that you will start to look at spheres and the curves on spheres as 3-dimensional objects. Imagining that you are a 2-dimensional bug walking on a sphere helps you to shed your limiting extrinsic 3-dimensional vision of the curves on a sphere and to experience straightness intrinsically. Ask yourself

- What does the bug have to do, when walking on a non-planar surface, in order to walk in a straight line?
- How can the bug check if it is going straight?

Experimentation with models plays an important role here. Working with models that *you create* helps you to experience great circles as, in fact, the only straight lines on the surface of a sphere. Convincing yourself of this notion will involve recognizing that straightness on the plane and straightness on a sphere have common elements. When you are comfortable with "great-circle-straightness," you will be ready to transfer the symmetries of straight lines on the plane to great circles on a sphere and, later, to geodesics on other surfaces. Here are some activities that you can try, or visualize, to help you experience great circles and their intrinsic straightness on a sphere. However, it is better for you to come up with your own experiences.

- Stretch something elastic on a sphere. It will stay in place on a great circle, but it will not stay on a small circle if the sphere is slippery. Here, the elastic follows a path that is approximately the shortest because a stretched elastic always moves so that it will be shorter. This a very useful practical criterion of straightness.
- Roll a ball on a straight chalk line (or straight on a freshly painted floor!). The chalk (or paint) will mark the line of contact on the sphere, and it will form a great circle.
- Take a stiff ribbon or strip of paper that does not stretch, and lay it "flat" on a sphere. It will only lie properly along a great circle. Do you see how this property is related to local symmetry? This is sometimes called the *Ribbon Test*. (For further discussion of the Ribbon Test, see Problems **3.4** and **7.6** of [**DG**: Henderson].)
- The feeling of turning and "non-turning" comes up. Why is it that on a great circle there is no turning and on a latitude line there is turning? Physically, in order to avoid turning, the bug has to move its left feet the same distance as its right feet. On a non-great circle (for example, a latitude line that is not the equator), the bug has to walk faster with the legs that are on the side closer to the equator. This same idea can be experienced by taking a small toy car with its wheels fixed so that, on a plane, it rolls along a straight line. On a sphere, the car will roll around a great circle; but it will not roll around other curves.

- Also notice that, on a sphere, straight lines are circles (points on the surface a fixed distance away from a given point) — special circles whose circumferences are straight! Note that the equator is a circle with two intrinsic centers: the north pole and the south pole. In fact, any circle (such as a latitude circle) on a sphere has two intrinsic centers.

These activities will provide you with an opportunity to investigate the relationships between a sphere and the geodesics of that sphere. Along the way, your experiences should help you to discover how great circles on a sphere have most of the same symmetries as straight lines on a plane.

You should pause and not read further until you have expressed your thinking and ideas about this problem.