Chapter 8

PARALLEL TRANSPORT

> ***Parallel*** straight lines are straight lines lying in a plane which do not meet if continued indefinitely in both directions.
> — Euclid, *Elements*, Definition 23 [Appendix A]

In this chapter we will further develop the notion of *parallel transport* that was introduced in Chapter 7.

PROBLEM 8.1 EUCLID'S EXTERIOR ANGLE THEOREM (EEAT)

> *Any exterior angle of a triangle is* greater *than each of the opposite interior angles.*[†]

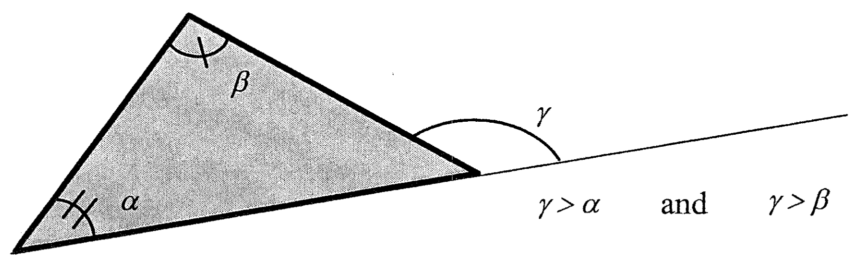

Figure 8.1 EEAT

[†]**Warning**: Euclid's EAT is not the same as the Exterior Angle Theorem usually studied in high school.

Look at EEAT on the plane, on a sphere, and on a hyperbolic plane.

SUGGESTIONS

You may find the following hint (which is found in Euclid's writings) useful: Draw a line from the vertex of α to the midpoint, M, of the opposite side, BC. Extend that line beyond M to a point A' in such a way that $AM \cong MA'$. Join A' to C. This hint will be referred to as *Euclid's hint*, and is pictured in Figure 8.2.

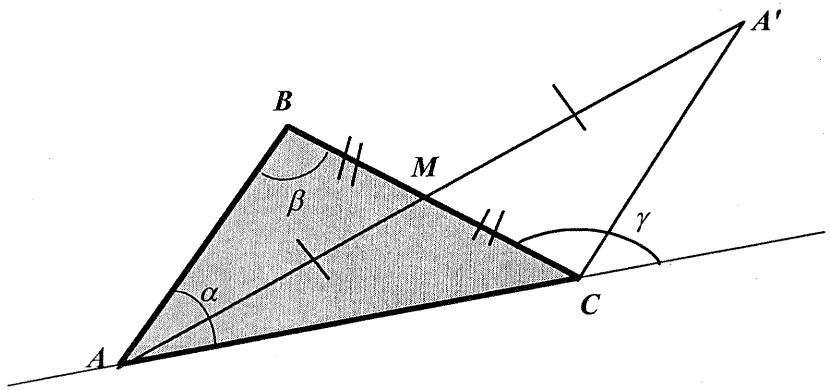

Figure 8.2 Euclid's Hint

Be cautious transferring this hint to a sphere. It will probably help to draw Euclid's hint directly on a physical sphere.

It is not necessary to use Euclid's hint to prove EEAT, and in fact many people don't "see" the hint. Another perfectly good way to prove EEAT is to use Problem **8.2**. Problems **8.1** and **8.2** are very closely related, and they can be done in either order. It is also fine to use **8.1** to prove **8.2** or use **8.2** to prove **8.1**, but of course don't do both. As a final note, remember you do not have to look at figures using only one orientation — rotations and reflections of a figure do not change its properties, so if you have trouble "seeing" something, check to see if it's something you're familiar with by orienting it differently on the page.

EEAT is not always true on a sphere, even for small triangles. Look at a counterexample as depicted in Figure 8.3. Then look at your proof of EEAT on the plane. It is very likely that your proof uses properties of angles and triangles that are true for small triangles on the sphere. Thus it may appear to you that your planar proof is also a valid proof of EEAT for small triangles on the sphere. But, there is a counterexample.

This could be, potentially, a very creative situation for you — **whenever you have a proof and counterexample of the same result, you have an opportunity to learn something deep and meaningful**. So, try out your planar proof of EEAT on the counterexample in Figure 8.3 and see what happens. Then try it on both large and small spherical triangles. If you can determine exactly which triangles satisfy EEAT and which triangles don't satisfy EEAT, then this information will be useful (but not crucial) to you in later problems.

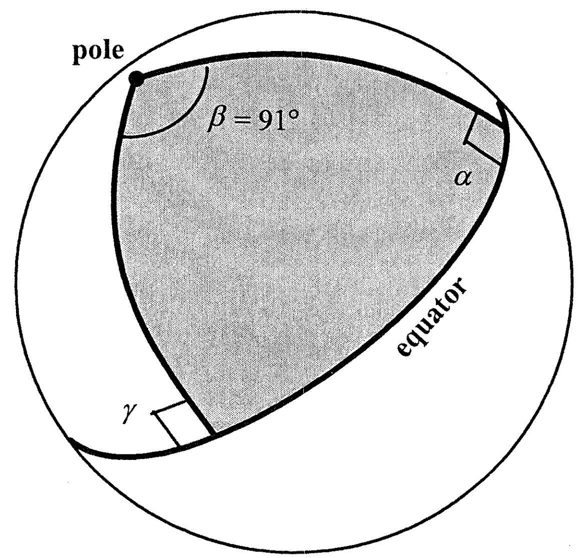

Figure 8.3 Counterexample to EEAT on a sphere

PROBLEM 8.2 SYMMETRIES OF PARALLEL TRANSPORTED LINES

*Consider two lines, **r** and **r'**, that are parallel transports of each other along a third line, **l**. Consider now the geometric figure that is formed by the three lines, one of them being a transversal to the other two, and look for the symmetries of that geometric figure.*

What can you say about the lines **r** *and* **r'**? *Do they intersect? If so, where? Look at the plane, spheres, and hyperbolic planes.*

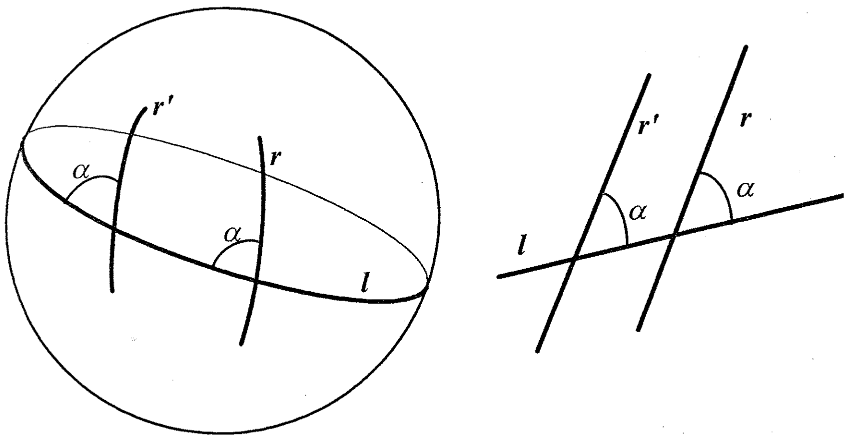

Figure 8.4a What can you say about ***r*** and ***r'***?

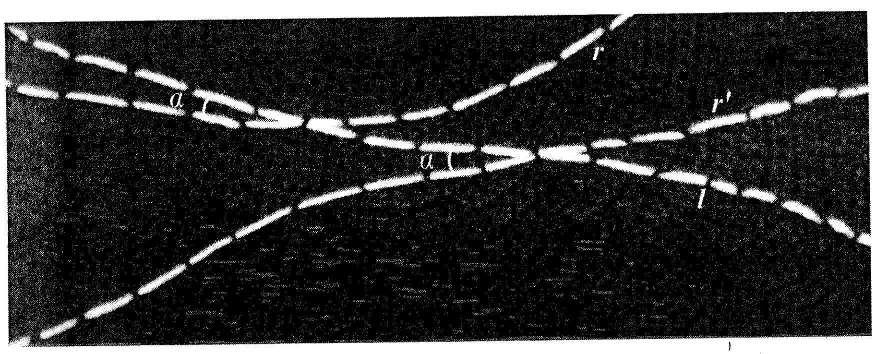

Figure 8.4b What can you say about ***r*** and ***r'***?

Suggestions

Parallel transport was already *informally* introduced in Chapter 7. In Problem **8.2** you have an opportunity to explore the concept further and prove its implications on the plane and a sphere. You will study the relationship between parallel transport and parallelism, as well.

A common high school definition of parallel lines is something like "two lines in a plane that never meet." But this is an inhuman definition — there is no way to check all points on both lines to see if they ever meet. This definition is also irrelevant on a sphere because we know that all geodesics on a sphere *will* cross each other. But we can measure the angles of a transversal. This is why it is more useful to talk about lines as parallel transports of one another rather than as parallel. So the question becomes:

> *If a transversal cuts two lines at congruent angles, are the lines in fact parallel in the sense of not intersecting?*

There are many ways to approach this problem. First, be sure to look at the symmetries of the local portion of the figure formed by the three lines. See what you can say about global symmetries from what you find locally. For the question of parallelism, you can use EEAT, but not if you used this problem to prove EEAT previously. Also, don't underestimate the power of symmetry when considering this problem. Many ideas that work on the plane will also be useful on a sphere and a hyperbolic plane, so try your planar proof on a sphere and a hyperbolic plane before attempting something completely different.

What is meant by *symmetry* with regard to geometric figures? A transformation is a *symmetry* of a geometric figure if it transforms that figure into itself. That is, the figure looks the same before and after the transformation. Here, we are looking for the symmetries of the plane and sphere and hyperbolic plane.

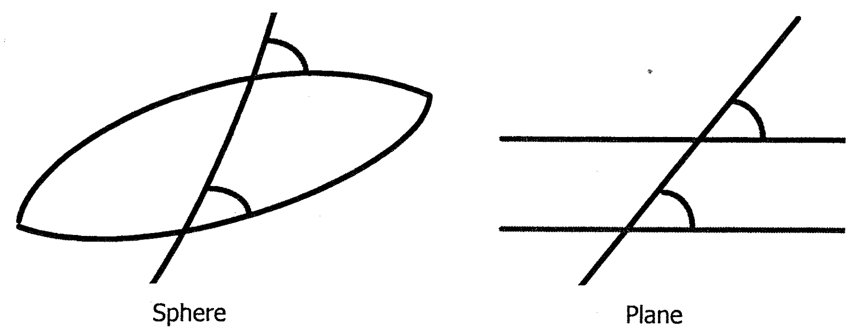

Figure 8.5 What are the symmetries?

From Figure 8.5, we can see that on a sphere we are looking for the symmetries of a *lune* cut at congruent angles by a geodesic. A ***lune*** is a spherical region bounded by two half great circles.

You may be inclined to use one or both of the following results: *Any transversal of a pair of parallel lines cuts these lines at congruent angles* (Problem **10.1**). And, *the angles of any triangle add up to a straight angle* (Problem **10.4**). The use of these results should be avoided for now, as they are both false on both a sphere and a hyperbolic plane. We have been investigating what is common between the plane, spheres, hyperbolic planes — trying to use common proofs whenever possible. In addition, we will find that it is necessary to make further assumptions about the plane before we can prove these results, and no additional assumptions are needed for Problem **8.2**. You may be tempted to use other properties of parallel lines that seem familiar to you, but in each case ask yourself whether or not the property is true on a sphere and on a hyperbolic plane. If it is not true on these surfaces, then don't use it here because it is not needed.

PROBLEM 8.3 TRANSVERSALS THROUGH A MIDPOINT

a. Prove: *If two geodesics **r** and **r'** are parallel transports along another geodesic **l**, then they are also parallel transports along any geodesic passing through the midpoint of the segment of **l** between **r** and **r'**. Does this hold for the plane, spheres, and hyperbolic planes?*

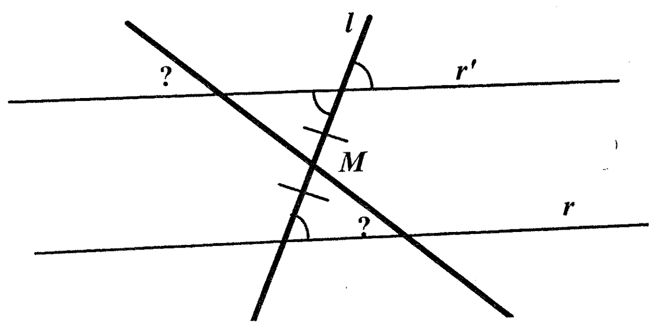

Figure 8.6 Transversals through a midpoint

b. *On a sphere or hyperbolic plane, are these the only lines that will cut **r** and **r'** at congruent angles? Why?*

c. Prove: *Two geodesics (on the plane, spheres, or hyperbolic planes) are parallel transports of each other if and only if they have a common perpendicular. On spheres and hyperbolic planes the common perpendicular is unique.*

All parts of this problem continue the ideas presented in Problem **8.2**. In fact, you may have proven this problem while working on **8.2** without even knowing it. There are many ways to approach this problem. Using symmetry is always a good way to start. You can also use some of the triangle congruence theorems that you have been working with. Look at the things you have discovered about transversals from Problems **8.1** and **8.2**; they are very applicable here. For the hyperbolic plane you may want to use results from Chapters 5 and/or 7.

PROBLEM 8.4 WHAT IS "PARALLEL"?

Since Chapter 7, you have been dealing with issues of parallelism. Parallel transport gives you a way to check parallelism. Even though parallel transported lines intersect on the sphere, there is a *feeling of local parallelness* about them. In most applications of parallel lines the issue is not whether the lines ever intersect, but whether a transversal intersects them at congruent angles at certain points; that is, whether the lines are *parallel transports of each other along the transversal.* You may choose to avoid definitions of parallel that do not give you a direct method of verification, such as these common definitions for parallel lines in the plane:

1. *Parallel lines are lines that never intersect;*
2. *Parallel lines are lines such that any transversal cuts them at congruent angles*; or,
3. *Parallel lines are lines that are everywhere equidistant.*

a. *Check out for each of these three definitions whether they apply to parallel transported lines on a sphere or on a hyperbolic plane.*

This is closely related to Problems **8.2** and **8.3**.